Hashing with Elliptic Curve *L*-Functions

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Plan

Introduction

Secure communication needs :

- Authenticity
- Integrity of information
- Non Repudiation

For this purpose :

- Cryptographic tools are precious :
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3. Given a message X (but with unknown K), it must be 'hard' to determine $h(K;X)$. Even when a large set of pairs $X_i; h(K;X_i)$ is known, it is 'hard' to determine the key K or to compute $h(K;X')$ for any new message $X' \neq X_i, \forall i$.

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Construction of hash function

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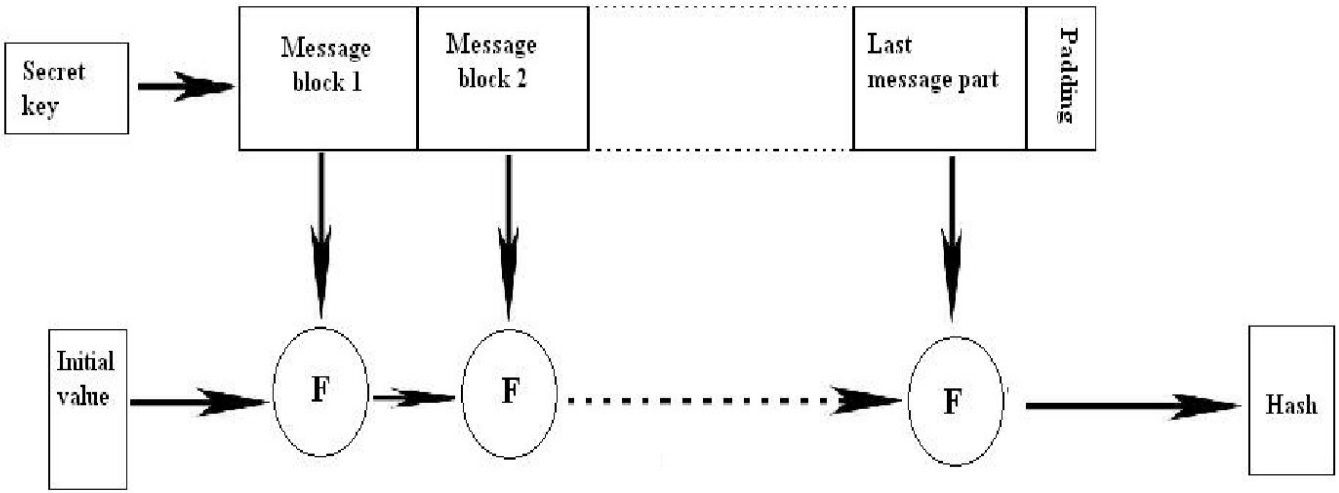
- Merkle-Damgard propose an iterative way to construct hash function from collision-resistant one-way compression functions.
- This construction was used in the design of many popular hash algorithms such as MD5, SHA1 and SHA2.



Elliptic curve

- An elliptic curve over $\mathbb{K}$ is defined by the equation

$$E : y^2 = x^3 + ax + b, \qquad a, b \in \mathbb{K},$$

- The discriminant $\Delta(E)$ of E is defined by $\Delta(E) = -4a^3 - 27b^2$.

Mordell-Weil theorem

$$E(\mathbb{K}) = \{(x, y) \in \mathbb{K} \times \mathbb{K} | y^2 = x^3 + ax + b\} \cup O$$

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Elliptic Curve *L*-Function

- In elliptic curve cryptography, we are interested in case where $\mathbb{K}$ is a finite field, $\mathbb{K} = \mathbb{F}_q$, q is a prime number power.
- For each prime p, we define

$$a_p = p + 1 - \sharp E(\mathbb{F}_p)$$

where $\mathbb{F}_p$ is a finite field with p elements and $\sharp E(\mathbb{F}_p)$ is the order of $E(\mathbb{F}_p) = O \cup \{(x, y) \in \mathbb{F}_p \times \mathbb{F}_p | y^2 = x^3 + ax + b\}$

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- The L-function attached to E is defined as

$$L(s,E) = \prod_{p|\Delta(E)} \frac{1}{1 - a_p p^{-s}} \prod_{p\nmid\Delta(E)} \frac{1}{1 - a_p p^{-s} + p^{1-2s}}.$$

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- ECOH divides the message into blocks, maps each block to an elliptic curve point and adds these points together with two more points. One additional point depends on the message length and the exclusive-or of all message blocks.
- given n message blocks $M_0, M_1, \ldots, M_{n-1}$

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- $$X_1 := P'(n), \quad X_2 := P''(\bigoplus_{i=0}^{n-1} M_i, n)$$
  - P' computes the padding point which depends only on the length of M.
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Conjecture

The following map

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where $L(s,E) = \sum_{n=1}^{\infty} a_n n^{-s}$, is a one way function.

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Authentication

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MAC's elliptic curve L-function protocol

1. Given a secret key (E, k, π) and a message $x = x_1 x_2 \ldots x_{t+1}$ where we divide the message x into $t+1$ blocks with size length r, k is a positive number and π is a permutation.
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Theorem

For any $(c_m, c_{m+1}, \ldots, c_{m+k}) \in \mathbb{Q}^{k+1}$ *satisfying* $|c_n| \leq \sqrt{n}\, d(n)$, *here* $d(n)$ *is the number of positive divisors of n. And for any* $\varepsilon > 0$, *there exits an algorithm which computes an elliptic curve E in* $\mathcal{L}_C^T$ *such that* $a_p = c_p$ *for all primes* $m \leq p \leq m + k$ *with a running time of order* $(CT)^{\frac{3}{2}+\varepsilon}$.

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